SOME EXAMPLES OF TILT-STABLE OBJECTS ON THREEFOLDS

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Abstract. We investigate properties and describe examples of tilt-stable objects on a smooth complex projective threefold. We give a structure theorem on slope semistable sheaves of vanishing discriminant, and describe certain Chern classes for which every slope semistable sheaf yields a Bridgeland semistable object of maximal phase. Then, we study tilt stability as the polarisation ω gets large, and give sufficient conditions for tilt-stability of sheaves of the following two forms: 1) twists of ideal sheaves or 2) torsion-free sheaves whose first Chern class is twice a minimum possible value.

Contents

1. Introduction

Let X be a smooth projective threefold over $\mathbb{C}$ throughout, unless otherwise stated. It has been a long standing open problem to construct a Bridgeland stability condition on an arbitrary Calabi-Yau threefold. In [BMT], this problem is reduced to showing a Bogomolov-Gieseker type inequality involving ch_3 for a class of objects they call tilt-stable objects. And in [BMT] and [Mac], this conjecture is proven for $X = \mathbb{P}^3$. The purpose of this paper is to give some examples of tilt-stable objects. There are at least two possible uses of specific examples of tilt stable objects: first to investigate the ch_3 bound conjectured in [BMT], and second, for understanding moduli spaces of Bridgeland stable objects.

We now give some details of the constructions introduced in [BMT]. Let ω, B be two numerical equivalence classes of $\mathbb{Q}$-divisors on X, with ω an ample class. Motivated by formulas for central charges arising in string theory, one defines a function $Z_{\omega,B} : D^b(X) \to \mathbb{C}$ on the bounded derived category $D^b(X)$ of coherent sheaves on X by

$$Z_{\omega,B}(E) = -\int_X e^{-B-i\omega}\mathrm{ch}(E) \tag{1.1}$$

$$= (-\widetilde{ch}_3(E) + \frac{\omega^2}{2}\widetilde{ch}_1(E)) + i(\omega\widetilde{ch}_2(E) - \frac{\omega^3}{6}\widetilde{ch}_0(E)) \tag{1.2}$$

where $\widetilde{ch}$ denotes the twisted Chern character $\widetilde{ch}(E) = e^{-B}ch(E)$. In [BMT], the function $Z_{\omega,B}$, along with an abelian category $\mathcal{A}_{\omega,B}$ that is the heart of a t-structure on $D^b(X)$, is conjectured to form a Bridgeland stability condition on $D^b(X)$, for any smooth projective threefold X over $\mathbb{C}$.

The heart $\mathcal{A}_{\omega,B}$ is constructed by a sequence of two tilts, starting with the abelian category $\mathrm{Coh}(X)$. After a tilt of $\mathrm{Coh}(X)$, the paper [BMT] defines a slope function $\nu_{\omega,B}$ on the resulting heart $\mathcal{B}_{\omega,B}$and says an object in $\mathcal{B}_{\omega,B}$ is "tilt-(semi)stable" if it is $\nu_{\omega,B}$-(semi)stable.

We now describe the results in this article. In Section 3, we show that if $E \in \mathcal{B}_{\omega,B}$ is a $\nu_{\omega,B}$-semistable object with $\nu_{\omega,B}(E) < \inf$, then $H^{-1}(E)$ must be a reflexive sheaf (Proposition 3.1). This allows us to use results on reflexive sheaves in studying tilt-semistable objects. For $E \in D^b(X)$, we can consider the discriminant in the sense of Drézet: $\overline{\Delta}_\omega(E) := (\omega^2\widetilde{ch}_1(E))^2 - 2(\omega^3\widetilde{ch}_0(E))(\omega\widetilde{ch}_2(E))$. In [BMT, Proposition 7.4.1], it is shown that if E is a slope-stable vector bundle on X with $\overline{\Delta}_\omega(E) = 0$, then E is tilt-stable. We show a partial converse to this:

Theorem 3.10. *Suppose $E \in \mathcal{B}_{\omega,B}$ satisfies all of the following three conditions:*
1. *$H^{-1}(E)$ is nonzero, torsion-free, $\mu_{\omega,B}$-stable (resp. $\mu_{\omega,B}$-semistable), with $\omega^2\widetilde{ch}_1(H^{-1}(E)) < 0$;*
2. *$H^0(E) \in \mathrm{Coh}^{\leq 1}(X)$;*
3. *$\overline{\Delta}_\omega(E) = 0$.*

Then E is tilt-stable (resp. tilt-semistable) if and only if $E = H^{-1}(E)[1]$ where $H^{-1}(E)$ is a locally free sheaf.

Using the above theorem, we also obtain a better understanding of slope semistable sheaves of zero discriminant:

Theorem 3.14. *Suppose $B = 0$. Let F be a μ_ω-semistable sheaf with $\overline{\Delta}_\omega(F) = 0$. Then $\mathscr{E}xt^1(F, \mathscr{O}_X)$ is zero, and F^* is locally free. Therefore, F is locally free if and only if the 0-dimensional sheaf $\mathscr{E}xt^2(F, \mathscr{O}_X)$ is zero.*

As a corollary, we show how every slope semistable sheaf of zero discriminant and zero tilt-slope yields a $Z_{\omega,0}$-semistable object of maximal phase in $\mathcal{A}_{\omega,0}$:

Theorem 3.17. *Suppose F is a μ_ω-semistable sheaf with $\overline{\Delta}_\omega(F) = 0$, $\nu_\omega(F) = 0$ and $\omega^2 ch_1(F) > 0$. Then $F^\vee[2]$ is an object of phase 1 with respect to $Z_{\omega,0}$ in $\mathcal{A}_{\omega,0}$.*

Since taking derived dual and shift both preserve families of complexes, Theorem 3.17 implies that the moduli of $Z_{\omega,0}$-semistable objects in $\mathcal{A}_{\omega,0}$ with the prescribed Chern classes (if it exists) contains the moduli of μ_ω-semistable sheaves as an open subspace. In the case of rank-one objects, for example, the open subspace contains the Hilbert scheme of points (see Remark 3.18).

In Section 4, we analyse tilt-stability at the large volume limit. In Remarks 3.11 and 4.2, we mention the connections between tilt-semistable objects and polynomial semistable objects.

In section 5, we give some sufficient conditions for a torsion-free sheaf $E \in \mathcal{T}_{\omega,B}$ with $\omega^2\widetilde{ch}_1(E) = 2c$ to be tilt-stable. Here, the number c is defined in [BMT, Lemma 7.2.2] as

$$c := \min\{\omega^2\widetilde{ch}_1(F) > 0 \mid F \in \mathcal{B}_{\omega,B}\}.$$

Tilt-semistable objects with $\omega^2\widetilde{ch}_1 = c$ were already characterised in [BMT]. Our results include:

Proposition 5.1. *Suppose $E \in \mathcal{T}_{\omega,B}$ is a torsion-free sheaf with $\nu_{\omega,B}(E) = 0$ and $\omega^2\widetilde{ch}_1(E) = 2c$, where c is defined above.*

(1) *If $\mu_{\omega,B,\max}(E) < \frac{\omega^3}{\sqrt{3}}$, then E is $\nu_{\omega,B}$-stable.*
(2) *If $\omega^3 > 3\omega(\widetilde{ch}_1(M))^2$ for every torsion free slope semistable sheaf M with $\omega^2\widetilde{ch}_1(M) = c$, then E is $\nu_{\omega,B}$-stable.*

We then apply this proposition to studying the tilt-stability of rank one torsion free sheaves that are twists of ideal sheaves of curves.

Finally, in Section 6, we use known inequalities between Chern characters of reflexive sheaves on $\mathbb{P}^3$ to describe many rank 3 slope-stable reflexive sheaves $E \in \mathcal{B}_{\omega,B}$ that are tilt-unstable. (An object $E \in \mathcal{B}_{\omega,B}$ is defined to be tilt-unstable if it is not tilt-semistable.) We give examples illustrating an observation in [BMT, p.4], that there are semistable sheaves on $\mathbb{P}^3$ with $\nu(E) = 0$ that do not satisfy $\widetilde{ch}_3(E) \leq \frac{\omega^2}{18}\widetilde{ch}_1(E)$ (the inequality in Conjecture 2.2). Since Conjecture 2.2 has been proven for $X = \mathbb{P}^3$ ([BMT], [Mac]), it follows that such E must be tilt-unstable. This shows that the notion of tilt-stability is a necessary hypothesis in Conjecture 2.2.

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Notation: We write $\mathrm{Coh}^{\leq i}(X) \subset \mathrm{Coh}(X)$ for the subcategory of coherent sheaves supported in dimension $\leq i$, and $\mathrm{Coh}^{\geq i+1}(X) \subset \mathrm{Coh}(X)$ for the subcategory of coherent sheaves that have no subsheaves supported in dimension $\leq i$.

2. Preliminaries

Throughout this article, X will always be a smooth projective threefold, unless otherwise specified.

In this section, we recall constructions introduced in [BMT]. Let us fix $\omega, B \in \mathrm{NS}(X)_{\mathbb{Q}}$ in the Neron-Severi group, with ω an ample class. The category $\mathcal{A}_{\omega,B}$ will be formed by starting with $\mathrm{Coh}(X)$ and tilting twice.

First, the twisted slope $\mu_{\omega,B}$ on $\mathrm{Coh}(X)$ is defined as follows. If $E \in \mathrm{Coh}(X)$ is a torsion sheaf, set $\mu_{\omega,B}(E) = +\infty$. Otherwise set

$$\mu_{\omega,B}(E) = \frac{\omega^2\widetilde{ch}_1(E)}{\widetilde{ch}_0(E)} = \frac{\omega^2(\mathrm{ch}_1(E) - B\mathrm{rk}(E))}{\mathrm{rk}(E)}. \tag{2.1}$$

Following [BMT, Section 3.1], we say $E \in \mathrm{Coh}(X)$ is $\mu_{\omega,B}$-(semi)stable if, for any $F \in \mathrm{Coh}(X)$ with $0 \neq F \subsetneq E$, we have $\mu_{\omega,B}(F) < (\leq) \mu_{\omega,B}(E/F)$. Let $\mu_\omega = \mu_{\omega,0}$. Since $\mu_{\omega,B}(E) = \mu_\omega(E) - B\omega^2$, it follows $E \in \mathrm{Coh}(X)$ is $\mu_{\omega,B}$-(semi)stable if and only if it is μ_ω-(semi)stable.

Let $\mathcal{T}_{\omega,B} \subset \mathrm{Coh}(X)$ be the category generated, via extensions, by $\mu_{\omega,B}$-semistable sheaves E of slope $\mu_{\omega,B}(E) > 0$, and let $\mathcal{F}_{\omega,B} \subset \mathrm{Coh}(X)$ be the subcategory generated by $\mu_{\omega,B}$-semistable sheaves of slope $\mu_{\omega,B} \leq 0$. Then $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ forms a torsion pair, and define the abelian category $\mathcal{B}_{\omega,B}$ as the tilt of $\mathrm{Coh}(X)$ with respect to $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$:

$$\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle.$$

For $E \in \mathcal{B}_{\omega,B}$, define its tilt-slope $\nu_{\omega,B}(E)$ as follows. If $\omega^2 \widetilde{ch}_1(E) = 0$, then set $\nu_{\omega,B}(E) = +\infty$. Otherwise set

$$\nu_{\omega,B}(E) = \frac{\Im Z_{\omega,B}(E)}{\omega^2 \widetilde{ch}_1(E)} = \frac{\omega \widetilde{ch}_2(E) - \frac{\omega^3}{6} \widetilde{ch}_0(E)}{\omega^2 \widetilde{ch}_1(E)}. \tag{2.2}$$

An object $E \in \mathcal{B}_{\omega,B}$ is defined to be $\nu_{\omega,B}$-(semi)stable if, for any non-zero proper subobject $F \subset E$ in $\mathcal{B}_{\omega,B}$, we have $\nu_{\omega,B}(F) < (\leq) \nu_{\omega,B}(E/F)$. We will use tilt-(semi)-stability and $\nu_{\omega,B}$-(semi)stability interchangably.

Let $\mathcal{T}'_{\omega,B}$ (resp. $\mathcal{F}'_{\omega,B}$) be the extension closed subcategory of $\mathcal{B}_{\omega,B}$ generated by $\nu_{\omega,B}$-stable objects $E \in \mathcal{B}_{\omega,B}$ of tilt-slope $\nu_{\omega,B}(E) > 0$ (resp. $\nu_{\omega,B}(E) \leq 0$). Then $(\mathcal{T}'_{\omega,B}, \mathcal{F}'_{\omega,B})$ form a torsion pair in $\mathcal{B}_{\omega,B}$, and tilting $\mathcal{B}_{\omega,B}$ with respect to $(\mathcal{T}'_{\omega,B}, \mathcal{F}'_{\omega,B})$ defines an abelian category $\mathcal{A}_{\omega,B} = \langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B} \rangle$.

In [BMT], it is shown that $\sigma = (Z_{\omega,B}, \mathcal{A}_{\omega,B})$ defines a Bridgeland stability condition as long as the image of the function $Z_{\omega,B}$ restricted to $\mathcal{A}_{\omega,B} \setminus \{0\}$ lies in the half-closed upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \Im z > 0, \text{or } [\Im z = 0 \text{ and } \Re z < 0]\}$. For $E \in \mathcal{A}_{\omega,B}$, it follows automatically from the construction of $\mathcal{A}_{\omega,B}$ that $\Im Z_{\omega,B}(E) \geq 0$; the difficulty so far is verifying that $\Re Z_{\omega,B}(E) < 0$ when $\Im Z_{\omega,B}(E) = 0$. To be more precise, in [BMT, Cor. 5.2.4], it is shown that $\sigma = (Z_{\omega,B}, \mathcal{A}_{\omega,B})$ is a Bridgeland stability condition on $D^b(X)$ if and only if the following conjecture holds:

Conjecture 2.1. [BMT, Conjecture 3.2.6] Any tilt-stable object $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$ satisfies

$$\widetilde{ch}_3(E) < \frac{\omega^2}{2} \widetilde{ch}_1(E). \tag{2.3}$$

In fact, an even stronger inequality is conjectured in [BMT]:

Conjecture 2.2. [BMT, Conjecture 1.3.1] Any tilt-stable object $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$ satisfies

$$\widetilde{ch}_3(E) \leq \frac{\omega^2}{18} \widetilde{ch}_1(E). \tag{2.4}$$

In [BMT] and [Mac], this conjecture is proven for $\mathbb{P}^3$, by using the fact that $\mathbb{P}^3$ has a full strong exceptional collection.

3. Reflexive sheaves and objects of zero discriminant

In [BMT, Proposition 7.4.1], it is shown that any slope stable vector bundle with zero discriminant is a tilt-stable object. The first goal of this section is to prove a partial converse to this result (Theorem 3.10). As a corollary, we produce a structure theorem on slope semistable sheaves of zero discriminant (Theorem 3.14). As another corollary, we show how, given any slope semistable sheaf of zero discriminant and $\nu_{\omega,B} = 0$, we can produce a $Z_{\omega,0}$-semistable object in $\mathcal{A}_{\omega,0}$ of

maximal phase (Theorem 3.17). This implies that the Hilbert scheme of points on X in contained in a moduli of Bridgeland semistable objects on X if the moduli exists (Remark 3.18).

We begin with the following link between reflexive sheaves and $\nu_{\omega,B}$-semistable objects in $\mathcal{B}_{\omega,B}$:

Proposition 3.1. *If $E \in \mathcal{B}_{\omega,B}$ is a $\nu_{\omega,B}$-semistable object with $\nu_{\omega,B}(E) < +\infty$, then $H^{-1}(E)$ is a reflexive sheaf.*

The proof of this proposition relies on:

Lemma 3.2. *Let $F \in \mathrm{Coh}(X)$ be a torsion-free sheaf, and let $F_n \in \mathrm{Coh}(X)$ be the Harder-Narasimhan $\mu_{\omega,B}$-semistable factor of F with greatest $\mu_{\omega,B}$-slope. If Q is the Harder-Narasimhan $\mu_{\omega,B}$-semistable factor of F^{**} with greatest $\mu_{\omega,B}$-slope, then $\mu_{\omega,B}(Q) = \mu_{\omega,B}(F_n)$. Hence if $F \in \mathcal{F}_{\omega,B}$, then $F^{**} \in \mathcal{F}_{\omega,B}$.*

Proof. Observe that F_n^{**} is a $\mu_{\omega,B}$-semistable sheaf, and we have a canonical inclusion $F_n^{**} \overset{\iota}{\hookrightarrow} F^{**}$. Hence $\mu_{\omega,B}(Q) \geq \mu_{\omega,B}(F_n^{**})$, by the proof of existence of HN filtrations. Let $Q \overset{\alpha}{\hookrightarrow} F^{**}$ be the inclusion, $F^{**} \overset{\beta}{\to} T$ be the cokernel of ι, and $K = \ker \beta\alpha$. We have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & K \arrow[r] \arrow[d, "k"] & F \arrow[d, "\iota"] \\
0 \arrow[r] & Q \arrow[r, "\alpha"] \arrow[d, "\beta\alpha"] & F^{**} \arrow[d, "\beta"] \\
 & T \arrow[r, "="] & T \arrow[d] \\
 & & 0
\end{tikzcd} \tag{3.1}$$

We have $\mu_{\omega,B}(K) = \mu_{\omega,B}(Q)$ (since $Q/K \subset T$ has codimension at least two), and $\mu_{\omega,B}(F_n) \geq \mu_{\omega,B}(K)$ (because $\mu_{\omega,B}(F_n)$ is greater than or equal to the slope of any subsheaf of F). Hence $\mu_{\omega,B}(F_n) \geq \mu_{\omega,B}(Q)$. Combined with $\mu_{\omega,B}(Q) \geq \mu_{\omega,B}(F_n^{**}) = \mu_{\omega,B}(F_n)$ we have $\mu_{\omega,B}(F_n) = \mu_{\omega,B}(Q)$.

The final statement follows from the definition that a sheaf $F \in \mathrm{Coh}(X)$ is in $\mathcal{F}_{\omega,B}$ if and only if $\mu_{\omega,B;\max}(F) := \mu_{\omega,B}(F_n) \leq 0$. □

Proof of Proposition 3.1. By Lemma 3.2, we have $H^{-1}(E)^{**} \in \mathcal{F}_{\omega,B}$. Hence the canonical short exact sequence $0 \to H^{-1}(E) \to H^{-1}(E)^{**} \to T \to 0$ gives us an injection $T \hookrightarrow H^{-1}(E)[1]$ in $\mathcal{B}_{\omega,B}$. Together with the injection $H^{-1}(E)[1] \hookrightarrow E$ in $\mathcal{B}_{\omega,B}$, we get $T \hookrightarrow E$ in $\mathcal{B}_{\omega,B}$ where $T \in \mathrm{Coh}^{\leq 1}(X)$. If $T \neq 0$, then $\nu_{\omega,B}(T) = \infty > \nu_{\omega,B}(E)$, contradicting the $\nu_{\omega,B}$-semistability of E. Hence $T = 0$, i.e. $H^{-1}(E)$ must be a reflexive sheaf. □

Corollary 3.3. *Let F be a torsion free sheaf with $F[1] \in \mathcal{B}_{\omega,B}$. If $F[1]$ is $\nu_{\omega,B}$-semistable, then F is reflexive.*

Lemma 3.4. *The subcategory* $\mathrm{Coh}^{\leq 0}(X)$ *of* $\mathcal{B}_{\omega,B}$ *is closed under quotients, subobjects and extensions.*

Proof. Given any short exact sequence $0 \to K \to Q \to B \to 0$ in $\mathcal{B}_{\omega,B}$ where $Q \in \mathrm{Coh}^{\leq 0}(X)$, consider the long exact sequence

$$0 \to H^{-1}(B) \to H^0(K) \to H^0(Q) \to H^0(B) \to 0.$$

If $H^{-1}(B)$ is nonzero, then it has positive rank, as does $H^0(K)$. However, then $0 < \omega^2\widetilde{ch}_1(H^0(K)) = \omega^2\widetilde{ch}_1(H^{-1}(B)) \leq 0$, which is a contradiction. Thus $H^{-1}(B) = 0$, and the lemma follows. $\square$

The next proposition roughly says that modifying an object in codimension 3 does not alter its $\nu_{\omega,B}$-(semi)stability:

Proposition 3.5. *Suppose we have a short exact sequence in* $\mathcal{B}_{\omega,B}$

$$0 \to E' \to E \to Q \to 0 \tag{3.2}$$

where $Q \in \mathrm{Coh}^{\leq 0}(X)$*.*

1. *If* E *is* $\nu_{\omega,B}$*-semistable (resp.* $\nu_{\omega,B}$*-stable), then* E' *is* $\nu_{\omega,B}$*-semistable (resp.* $\nu_{\omega,B}$*-stable).*
2. *Assuming* $\mathrm{Hom}(\mathrm{Coh}^{\leq 0}(X), E) = 0$*, if* E' *is* $\nu_{\omega,B}$*-semistable, then* E *is* $\nu_{\omega,B}$*-semistable.*
3. *Assuming* $\mathrm{Hom}(\mathrm{Coh}^{\leq 0}(X), E) = 0$ *and* $\omega^2\widetilde{ch}_1(E) \neq 0$*, if* E' *is* $\nu_{\omega,B}$*-stable then* E *is* $\nu_{\omega,B}$*-stable.*
4. *If* E *satisfies Conjecture 2.4 then* E' *also satisfies the same conjecture.*

Proof. Consider a commutative diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A' \arrow[r, "\alpha"] \arrow[d, "\gamma'"] & A \arrow[d, "\gamma"] & & \\
0 \arrow[r] & E' \arrow[r, "e"] \arrow[d, "\delta'"] & E \arrow[r] \arrow[d, "\delta"] & Q \arrow[r] & 0 \\
 & B' \arrow[r, "\beta"] \arrow[d] & B \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the row is the exact sequence (3.2), and both columns are short exact sequences in $\mathcal{B}_{\omega,B}$.

Proof of part 1. Suppose A' is a nonzero proper subobject of E'. We can put $A = A'$, $\alpha = \mathrm{id}_{A'}$, $\gamma = e\gamma'$, and let β be the induced map of cokernels from the upper commutative square. Then by the snake lemma in the abelian category $\mathcal{B}_{\omega,B}$, coker (β) is a quotient of Q in $\mathcal{B}_{\omega,B}$, and hence is a 0-dimensional sheaf by Lemma 3.4, while ker $(\beta) = 0$. Thus $\nu_{\omega,B}(B') = \nu_{\omega,B}(B)$. We also have $\nu_{\omega,B}(A') = \nu_{\omega,B}(A)$ (since $A' = A$). Note that A is a nonzero proper subobject of E. If E is $\nu_{\omega,B}$-semistable, then $\nu_{\omega,B}(A) \leq \nu_{\omega,B}(B)$, implying $\nu_{\omega,B}(A') \leq \nu_{\omega,B}(B')$, and hence E' is $\nu_{\omega,B}$-semistable. Similarly, if E is $\nu_{\omega,B}$-stable, then E' is also $\nu_{\omega,B}$-stable.

Proof of part 2. Suppose that A is a nonzero proper subobject of E. We can put $B' = \mathrm{im}\,(\delta e)$, $\delta' = \delta e$, $A' = \ker(\delta')$, put β as the canonical inclusion $\mathrm{im}\,(\delta') \hookrightarrow B$, and put α as the induced map of kernels from the lower commutative square. If $A' = 0$, then δ' is an isomorphism. However, this implies that δ restricts to an injection from E', i.e. $E' \cap A = 0$. Hence the quotient $E \twoheadrightarrow Q$ induces an injection $A \hookrightarrow Q$, so $A \in \mathrm{Coh}^{\leq 0}(X)$ by Lemma 3.4, which contradicts our assumption $\mathrm{Hom}(\mathrm{Coh}^{\leq 0}(X), E) = 0$. Therefore, A' is nonzero.

On the other hand, if $A' = E'$, then δ' is the zero map, meaning $E' \subset A$, and so there is a surjection $Q \twoheadrightarrow B$ in $\mathcal{B}_{\omega,B}$. By Lemma 3.4, $B \in \mathrm{Coh}^{\leq 0}(X)$, and hence $\nu_{\omega,B}(B) = \infty$. So $\nu_{\omega,B}(A) \leq \nu_{\omega,B}(B)$ when $A' = E'$.

Now, suppose A' is a nonzero proper subobject of E'. Since α, e and β are all injective maps, the snake lemma gives an induced short exact sequence in $\mathcal{B}_{\omega,B}$ of their cokernels:

$$0 \to \mathrm{coker}\,(\alpha) \to Q \to \mathrm{coker}\,(\beta) \to 0. \tag{3.3}$$

Hence $\mathrm{coker}\,(\alpha), \mathrm{coker}\,(\beta)$ are both 0-dimensional sheaves by Lemma 3.4, giving us $\nu_{\omega,B}(A') = \nu_{\omega,B}(A)$ and $\nu_{\omega,B}(B') = \nu_{\omega,B}(B)$. If E' is $\nu_{\omega,B}$-semistable, then $\nu_{\omega,B}(A') \leq \nu_{\omega,B}(B')$, implying $\nu_{\omega,B}(A) \leq \nu_{\omega,B}(B)$, and hence E is $\nu_{\omega,B}$-semistable.

Proof of part 3. The proof is essentially same as for part 2, with the following additional argument for the scenario $A' = E'$. If $A' = E'$, the hypothesis $0 \neq \omega^2\widetilde{ch}_1(E) = \omega^2\widetilde{ch}_1(E')$ along with the injection $E' \hookrightarrow A$ in $\mathcal{B}_{\omega,B}$ implies $\omega^2\widetilde{ch}_1(A) = \omega^2\widetilde{ch}_1(E') > 0$ and hence $\nu_{\omega,B}(A) < \infty = \nu_{\omega,B}(B)$.

Proof of part 4. Assume E is $\nu_{\omega,B}$-stable, $\nu_{\omega,B}(E) = 0$, and $\widetilde{ch}_3(E) \leq \frac{\omega^2}{18}\widetilde{ch}_1(E)$. Since the formula for $\nu_{\omega,B}$ does not have any dependence on $\widetilde{ch}_3$, we have $\nu_{\omega,B}(E) = \nu_{\omega,B}(E')$, so $\nu_{\omega,B}(E') = 0$. By part 1, E' is $\nu_{\omega,B}$-stable. Finally,

$$\widetilde{ch}_3(E') = \widetilde{ch}_3(E) - \widetilde{ch}_3(Q) \leq \widetilde{ch}_3(E) \leq \frac{\omega^2}{18}\widetilde{ch}_1(E) = \frac{\omega^2}{18}\widetilde{ch}_1(E').$$

□

Example 3.6. Let E be a $\mu_{\omega,B}$-stable vector bundle on X with $\overline{\Delta}_\omega(E) = 0$. Then E is $\nu_{\omega,B}$-stable by [BMT, Prop. 7.4.1]. Assume $\omega^2\widetilde{ch}_1(E) > 0$, so $E \in \mathcal{T}_{\omega,B}$. Begining with any surjection $E \twoheadrightarrow Q$ in $\mathrm{Coh}(X)$ with $Q \in \mathrm{Coh}^{\leq 0}(X)$ we can apply Proposition 3.5 to obtain other examples of tilt-stable objects. For example, suppose X has Picard number one. Then any line bundle L on X satisfies $\overline{\Delta}_\omega(L) = 0$. Choose a line bundle L with $\omega^2\widetilde{ch}_1(L) > 0$. Let I_Z be the ideal sheaf of any zero dimensional subscheme $Z \subseteq X$. Then applying Proposition 3.5 to the exact sequence $0 \to I_Z \otimes L \to L \to \mathcal{O}_Z \to 0$ shows $I_Z \otimes L$ is tilt-stable.

For objects $E \in D^b(X)$, we have the following two versions of discriminants (see [BMT, Section 7.3] for some background information):

(1) $\Delta(E) := (\widetilde{ch}_1(E))^2 - 2(\widetilde{ch}_0(E))(\widetilde{ch}_2(E))$, the definition that is usually used for coherent sheaves;
(2) $\overline{\Delta}_\omega(E) := (\omega^2\widetilde{ch}_1(E))^2 - 2(\omega^3\widetilde{ch}_0(E))(\omega\widetilde{ch}_2(E))$.

A calculation shows $\Delta(E) = (ch_1(E))^2 - 2(ch_0(E))(ch_2(E))$ that is, we may omit the tildes over the ch_i, and in particular the $\Delta(E)$ is independent of B. If the Picard number of X is one, then $\overline{\Delta}_\omega$ is independent of B [Mac, Section 2.1], but in general $\overline{\Delta}_\omega$ depends on B.

For later use, we will need the following lemma.

Lemma 3.7. *For any coherent sheaf F on X, we have $\overline{\Delta}_\omega(F) \geq (\omega\Delta(F))\omega^3$.*

Proof. The Hodge Index Theorem gives $(\omega^2\widetilde{ch}_1(F))^2 \geq (\omega^3)(\omega\widetilde{ch}_1(F)^2)$, and hence

$$\overline{\Delta}_\omega(F) = (\omega^2\widetilde{ch}_1(F))^2 - 2(\omega^3\widetilde{ch}_0(F))(\omega\widetilde{ch}_2(F)) \tag{3.4}$$

$$\geq \omega^3(\omega\widetilde{ch}_1(F)^2) - 2(\omega^3\widetilde{ch}_0(F))(\omega\widetilde{ch}_2(F)) = \omega^3(\omega\Delta(F)). \tag{3.5}$$

□

The following result was shown in [BMT, Cor 7.3.2], and it was a key ingredient for the main result in [Mac].

Proposition 3.8. [BMT, Cor 7.3.2] *If $E \in \mathcal{B}_{\omega,B}$ is $\nu_{\omega,B}$-semistable, then $\overline{\Delta}_\omega(E) \geq 0$.*

In this section, we will investigate the tilt-stability of objects with $\overline{\Delta}_\omega(E) = 0$. The following result gives many examples of tilt-stable objects. (Furthermore, in [BMT, Proposition 7.4.2], they verify these objects satisfy Conjecture 2.2, and equality holds).

Proposition 3.9. [BMT, Proposition 7.4.1] *Let E be a $\mu_{\omega,B}$-stable vector bundle on X with $\overline{\Delta}_\omega(E) = 0$. Then E is $\nu_{\omega,B}$-stable.*

Now we come to the following partial converse to Proposition 3.9.

Theorem 3.10. *Suppose $E \in \mathcal{B}_{\omega,B}$ satisfies all of the following three conditions:*

1. *$H^{-1}(E)$ is nonzero, torsion-free, $\mu_{\omega,B}$-stable (resp. $\mu_{\omega,B}$-semistable), with $\omega^2\widetilde{ch}_1(H^{-1}(E)) < 0$;*
2. *$H^0(E) \in \mathrm{Coh}^{\leq 1}(X)$;*
3. *$\overline{\Delta}_\omega(E) = 0$.*

Then E is tilt-stable (resp. tilt-semistable) if and only if $E = H^{-1}(E)[1]$ where $H^{-1}(E)$ is a locally free sheaf.

Remark 3.11. Note that, any polynomial stable complex on X that is PT-semistable or dual-PT-semistable (see [Lo2]) of positive degree satisfies conditions (1) and (2) in Theorem 3.10. However, the theorem says that, under the assumption $\overline{\Delta}_\omega = 0$, a (dual-)PT-semistable object cannot be a genuine complex if it is to be tilt-semistable.

We break up the proof of Theorem 3.10 into a couple of intermediate results.

Proposition 3.12. *Let F be a $\mu_{\omega,B}$-semistable reflexive sheaf on X such that $\overline{\Delta}_\omega(F) = 0$. Then F is a locally free sheaf.*

Proof. The proof is largely based on that of [BMT, Proposition 7.4.2]. By [Laz, Theorem 4.1.10], we can find a pair (f, L) where f is a morphism $Y \to X$ that is finite, surjective and flat, with Y a smooth projective variety, and a line bundle L on Y such that $(f^*\omega)^2\widetilde{ch}_1(L \otimes f^*F) = 0$.

Since f is flat and both X, Y are smooth, $L \otimes f^*F$ is reflexive by [Har, Proposition 1.8]. On the other hand, by choosing L above so that $c_1(L)$ is a rational multiple of $f^*\omega$, we have $\overline{\Delta}_{f^*\omega}(L \otimes f^*F) = \overline{\Delta}_{f^*\omega}(f^*F) = 0$ because the discriminant $\overline{\Delta}_{f^*\omega}$ is invariant under tensoring by a line bundle whose c_1 is proportional to $f^*\omega$, and $\overline{\Delta}_\omega(F) = 0$. Hence $(f^*\omega)\widetilde{ch}_2(L \otimes f^*F) = 0$. Passing to another finite cover of the form above, we can assume that B is the divisor class of a line bundle M

on Y, and $ch(M \otimes L \otimes f^*F) = \widetilde{ch}(L \otimes f^*F)$. Now, f^*F is $\mu_{f^*\omega, f^*B}$-semistable since f is a finite morphism. Hence $M \otimes L \otimes f^*F$ is $\mu_{f^*\omega, f^*B}$-semistable (and equivalently, $\mu_{f^*\omega}$-semistable) with vanishing $(f^*\omega)^2 ch_1$ and $(f^*\omega) ch_2$. Thus, by [Lan2, Proposition 5.1], $M \otimes L \otimes f^*F$ is locally free, i.e. f^*F is locally free. Since f is surjective and flat, it is faithfully flat, and so F itself is locally free. □

Lemma 3.13. *If $E \in \mathcal{B}_{\omega,B}$ with $H^{-1}(E)$ a vector bundle, and $H^0(E) \in \mathrm{Coh}^{\leq 0}(X)$, then $E \cong H^{-1}(E)[1] \oplus H^0(E)$. If E further satisfies $\mathrm{Hom}(\mathrm{Coh}^{\leq 0}(X), E) = 0$ or E is $\nu_{\omega,B}$-stable, then $H^0(E) = 0$, in which case $E \simeq H^{-1}(E)[1]$ is a shift of a vector bundle.*

Proof. Let $F = H^{-1}(E)$ and $T = H^0(E)$. We have $\mathrm{Ext}^1(T, F[1]) = \mathrm{Ext}^2(T, F) = \mathrm{Ext}^1(F, T \otimes \omega_X) = H^1(X, F^* \otimes T \otimes \omega_X)$, which is zero since $T \in \mathrm{Coh}^{\leq 0}(X)$. From the exact sequence $F[1] \to E \to T$ in $\mathcal{B}_{\omega,B}$ we conclude $E \simeq F[1] \oplus T$. If E is $\nu_{\omega,B}$-stable, then $T = 0$ (otherwise T would be a $\nu_{\omega,B}$-destabilizing object of E). □

Proof of Theorem 3.10. If $E = H^{-1}(E)[1]$ where $H^{-1}(E)$ is a $\mu_{\omega,B}$-stable (resp. $\mu_{\omega,B}$-semistable) locally free sheaf satisfying (1) through (3), then the result is [BMT, Proposition 7.4.1]. (Note that, [BMT, Proposition 7.4.1] still holds if we replace each occurence of 'stable' by 'semistable' in its statement.)

Now, assume E satisfies (1) through (3) and is tilt-semistable. Let $F = H^{-1}(E)$. Then by Proposition 3.1, F is reflexive. The condition $H^0(E) \in \mathrm{Coh}^{\leq 1}(X)$ implies $\omega^3 \widetilde{ch}_0(H^0(E)) = \omega^2 \widetilde{ch}_1(H^0(E)) = 0$, and hence the condition $\overline{\Delta}_\omega(E) = 0$ can be rewritten as

$$\overline{\Delta}_\omega(F) + 2\omega^3 \widetilde{ch}_0(F) \omega \widetilde{ch}_2(H^0(E)) = 0. \tag{3.6}$$

The Bogomolov-Gieseker inequality says $\omega\Delta(F) \geq 0$, and hence by Lemma 3.7, we have $\overline{\Delta}_\omega(F) \geq 0$. Since both terms $\overline{\Delta}_\omega(F)$ and $2\omega^3 \widetilde{ch}_0(F) \omega \widetilde{ch}_2(H^0(E))$ are nonnegative, Equation 3.6 implies they must both by zero. So $\omega \widetilde{ch}_2(H^0(E)) = 0$, and $H^0(E) \in \mathrm{Coh}^{\leq 0}(E)$. Since $\overline{\Delta}_\omega(F) = 0$, we have F is locally free by Proposition 3.12. By Lemma 3.13, we can conclude $E \simeq F[1]$. □

Using Theorem 3.10, we can also prove the following result on μ_ω-semistable sheaves of zero discriminant:

Theorem 3.14. *Suppose $B = 0$. Let F be a μ_ω-semistable torsion-free sheaf with $\overline{\Delta}_\omega(F) = 0$. Then $\mathscr{E}xt^1(F, \mathscr{O}_X)$ is zero, and F^* is locally free. Therefore, F is locally free if and only if the 0-dimensional sheaf $\mathscr{E}xt^2(F, \mathscr{O}_X)$ is zero.*

To prove Theorem 3.14, we first note:

Lemma 3.15. *Suppose $B = 0$. If F is a μ_ω-semistable torsion-free sheaf on X with $\overline{\Delta}_\omega(F) = 0$, then F must be locally free outside a codimension-3 locus.*

Proof. Suppose the singularity locus of F has codimension 2. Then $\widetilde{ch}_2(F^{**}/F) = ch_2(F^{**}/F) > 0$, implying $\overline{\Delta}_\omega(F^{**}) < \overline{\Delta}_\omega(F) = 0$, which is a contradiction by Lemma 3.7 and the usual Bogomolov-Gieseker inequality for μ_ω-semistable sheaves. Hence the singularity locus of F has codimension at least 3. □

Lemma 3.16. *Suppose $B = 0$. Suppose F is a μ_ω-semistable (resp. μ_ω-stable) torsion-free sheaf, such that $\omega^2 ch_1(F) > 0$ and $\overline{\Delta}_\omega(F) = 0$. Then $(\tau^{\leq 1} F^\vee)[1]$ is a $\nu_{\omega,0}$-semistable (resp. $\nu_{\omega,0}$-stable) object.*

Proof. By Lemma 3.15, the sheaf F is locally free outside a 0-dimensional locus. Hence $\mathscr{E}xt^i(F, \mathscr{O}_X)$ is 0-dimensional for all $i > 0$, implying $\overline{\Delta}_\omega(F^*) = 0$. Since F^* is reflexive, Proposition 3.12 implies F^* is locally free. And so $F^*[1]$ is $\nu_{\omega,0}$-semistable by [BMT, Proposition 7.4.1]. Applying $\mathrm{Hom}(\mathrm{Coh}^{\leq 0}(X), -)$ to the exact triangle in $D(X)$

$$\tau^{\geq 2}(F^\vee) \to (\tau^{\leq 1}(F^\vee))[1] \to F^\vee[1] \to \tau^{\geq 2}(F^\vee)[1] \tag{3.7}$$

and writing $E := (\tau^{\leq 1}(F^\vee))[1]$, we obtain $\mathrm{Hom}(\mathrm{Coh}^{\leq 0}(X), E) = 0$. Hence, by applying Proposition 3.5 to the short exact sequence

$$0 \to F^*[1] \to E \to \mathscr{E}xt^1(F, \mathscr{O}_X) \to 0$$

in $\mathcal{B}_{\omega,B}$, we get that E itself is $\nu_{\omega,0}$-semistable. □

We can now finish the proof of Theorem 3.14:

Proof of Theorem 3.14. By tensoring F with $\mathscr{O}_X(m\omega)$ for $m \gg 0$, we can assume $\omega^2\widetilde{ch}_1(F) > 0$. From the proof of Lemma 3.16, we know F^* is locally free. By Lemma 3.16 and Theorem 3.10, we have $H^0(\tau^{\leq 1}(F^\vee))[1] = 0$, i.e. $\mathscr{E}xt^1(F, \mathscr{O}_X)$ is zero. The last assertion of Theorem 3.14 follows from the fact that any torsion-free sheaf on a smooth threefold has homological dimension at most 2. □

Recall the following easy consequence of [BMT, Propositions 7.4.1, 7.4.2]: suppose F is a $\mu_{\omega,B}$-stable vector bundle on X with $\overline{\Delta}_\omega(F) = 0$ and $\nu_{\omega,B}(F) = 0$. Then the object $F[1]$ (resp. $F[2]$) lies in $\mathcal{A}_{\omega,B}$, has phase 1 with respect to $Z_{\omega,B}$ and hence is $Z_{\omega,B}$-semistable if $\omega^2\widetilde{ch}_1(F) > 0$ (resp. $\omega^2\widetilde{ch}_1(F) \leq 0$). Now we have a slight extension of this result:

Theorem 3.17. *Suppose F is a μ_ω-semistable sheaf with $\overline{\Delta}_\omega(F) = 0$, $\nu_\omega(F) = 0$ and $\omega^2 ch_1(F) > 0$. Then $F^\vee[2]$ is an object of phase 1 with respect to $Z_{\omega,0}$ in $\mathcal{A}_{\omega,0}$.*

In particular, if $(\mathcal{A}_{\omega,0}, Z_{\omega,0})$ is a stability condition, then we can speak of $F^\vee[2]$ as a $Z_{\omega,0}$-semistable object.

Proof. By Lemma 3.16, we know $(\tau^{\leq 1}F^\vee)[1]$ is $\nu_{\omega,0}$-semistable with $\nu_{\omega,0} = 0$. Hence $(\tau^{\leq 1}F^\vee)[1] \in \mathcal{F}'_{\omega,0}$, and so $(\tau^{\leq 1}F^\vee)[2] \in \mathcal{A}_{\omega,0}$. Since $(\tau^{\geq 2}F^\vee)[2]$ also lies in $\mathcal{A}_{\omega,0}$ and has phase 1 with respect to $Z_{\omega,0}$, from the exact triangle (3.7) we see that $F^\vee[2]$ is also of phase 1 in $\mathcal{A}_{\omega,0}$. □

Remark 3.18. Given Theorem 3.17, it is reasonable to hope that for any Chern character ch satisfying the conditions in the theorem, the moduli space of $Z_{\omega,0}$-semistable objects in $\mathcal{A}_{\omega,0}$ (provided $(\mathcal{A}_{\omega,0}, Z_{\omega,0})$ is a stability condition and the moduli space exists) contains the moduli of slope semistable sheaves of Chern character ch as a subspace.

More concretely, suppose $Z \subset X$ is a 0-dimensional subscheme of length n, and let L be a line bundle on X such that $I_Z \otimes L$ satisfies the hypotheses of Theorem 3.17. For instance, we can choose L so that $c_1(L)$ is proportional to ω (so that tensoring I_Z by L does not alter its $\overline{\Delta}_\omega$); on the other hand, it can be checked easily that $\nu_\omega(I_Z \otimes L) = 0$ is equivalent to $3\omega c_1(L)^2 = \omega^3$, provided $\omega^2 c_1(L) \neq 0$. Then $(I_Z \otimes L)^\vee[2]$ would be an object of of $\mathcal{A}_{\omega,0}$ with phase 1 with respect to $Z_{\omega,0}$, and hence would be $Z_{\omega,0}$-semistable in $\mathcal{A}_{\omega,0}$. Therefore, if the moduli space of $Z_{\omega,0}$-semistable objects $E \in \mathcal{A}_{\omega,0}$ with fixed chern character $ch(E) = ch((I_Z \otimes L)^\vee[2])$ exists, then it contains the Hilbert scheme of n points on X. The following lemma

shows that, under the condition $H^{-1}(E)=0$, a $Z_{\omega,0}$-semistable object $E\in\mathcal{A}_{\omega,0}$ with the same Chern classes as $(I_Z\otimes L)^\vee[2]$ is 'almost' (i.e. up to a 0-dimensional sheaf sitting at degree 0) of the form $(I_Z\otimes L)^\vee[2]$.

Lemma 3.19. *Suppose $B=0$, and any line bundle on X with the same Chern classes as $\mathscr{O}_X$ is isomorphic to $\mathscr{O}_X$ (e.g. when X has Picard rank 1). Suppose $E\in\mathcal{A}_{\omega,0}$ is such that $ch(E)=ch((I_Z\otimes L)^\vee[2])$ where I_Z, L are as in Remark 3.18. (In particular, this means $\omega^2 ch_1(E)\neq 0$, $\nu_{\omega,0}(E)=0$, and $Z_{\omega,0}(E)$ has phase 1.) If $H^{-1}(E)=0$, then $H^0(E^\vee[2])\cong I_Y\otimes L$ where I_Y is the ideal sheaf of some 0-dimensional subscheme Y of X, and $H^0(E)$ is a 0-dimensional sheaf.*

Proof. With respect to $Z_{\omega,0}$-stability, E has a filtration in $\mathcal{A}_{\omega,0}$ with $Z_{\omega,0}$-stable factors E^i. Since $\Im Z_{\omega,0}(E)=0$, the same holds for each E^i. For each i, we have a canonical short exact sequence in $\mathcal{A}_{\omega,0}$

$$0\to E_1^i[1]\to E^i\to E_2^i\to 0$$

where $E_1^i\in\mathcal{F}'_{\omega,0}$ and $E_2^i\in\mathcal{T}'_{\omega,0}$. Since E^i is $Z_{\omega,0}$-stable, for each i, either $E^i=E_1^i[1]$ or $E^i=E_2^i$.

We now make an observation on objects in $\mathcal{T}'_{\omega,0}$: Suppose G is any object in $\mathcal{T}'_{\omega,0}$ with $\Im Z_{\omega,0}(G)=0$. Then G is necessarily $Z_{\omega,0}$-semistable as an object in $\mathcal{A}_{\omega,0}$. With respect to $\nu_{\omega,0}$-stability, G has a filtration in $\mathcal{B}_{\omega,0}$ with $\nu_{\omega,0}$-stable factors G^i. By the definition of $\mathcal{T}'_{\omega,0}$, we know $\nu_{\omega,0}(G^i)>0$ for each i. On the other hand, each G^i lies in $\mathcal{T}'_{\omega,0}\subset\mathcal{A}_{\omega,0}$, and so G is an extension of the G^i in $\mathcal{A}_{\omega,0}$ as well. Hence $\Im Z_{\omega,0}(G^i)=0$ for all i. Now, if $\omega^2 ch_1(G^i)\neq 0$ for some i, then $\omega^2 ch_1(G^i)>0$, and so $\Im Z_{\omega,0}(G^i)>0$, which is a contradiction. Hence $\omega^2 ch_1(G^i)=0$ for all i. By [BMT, Remark 3.2.2], each G^i lies in the extension-closed category

$$\mathcal{C}:=\langle \mathrm{Coh}^{\leq 1}(X), F[1] : F\ \mu_{\omega,0}\text{-stable with } \mu_{\omega,0}(F)=0\rangle\subset\mathcal{B}_{\omega,0}.$$

Note that, every object in $\mathcal{C}$ has $\nu_{\omega,0}=+\infty$, and is thus $\nu_{\omega,0}$-semistable. Hence $\mathcal{C}\subset\mathcal{T}'_{\omega,0}$, and each G^i, being $\nu_{\omega,0}$-stable, either lies in $\mathrm{Coh}^{\leq 1}(X)$ or is of the form $F[1]$ for some $\mu_{\omega,0}$-stable sheaf of $\mu_{\omega,0}=0$. Furthermore, if G^i lies in $\mathrm{Coh}^{\leq 1}(X)$, then it must lie in $\mathrm{Coh}^{\leq 0}(X)$ since $\Im Z_{\omega,0}(G^i)=0$.

Now, from the canonical short exact sequence

$$0\to E_1[1]\to E\to E_2\to 0 \tag{3.8}$$

in $\mathcal{A}_{\omega,0}$, we see that $H^{-1}(E)=0$ implies $H^0(E_1)=0$ and $H^{-1}(E_2)=0$. That is, both E_1, E_2 are sheaves (up to shift). In particular, by our observation above, E_2 must be an extension of objects in $\mathrm{Coh}^{\leq 0}(X)$, and so $H^0(E)\cong E_2\in\mathrm{Coh}^{\leq 0}(X)$.

On the other hand, $H^{-1}(E_1)$ is a rank-one torsion-free sheaf by our assumption on $ch(E)$. Dualising (3.8) and shifting, we get an exact triangle

$$E_2^\vee[2]\to E^\vee[2]\to E_1^\vee[1]. \tag{3.9}$$

Since E_2 is a 0-dimensional sheaf at degree 0, $E_2^\vee[2]\in\mathrm{Coh}^{\leq 0}(X)[-1]$. On the other hand, since E_1 is a sheaf at degree -1, the complex $E_1^\vee[1]$ sits at degrees 0 through 3. The long exact sequence of cohomology of (3.9) then looks like

$$0\to H^0(E^\vee[2])\to H^{-1}(E_1)^*\to \mathscr{H}om(E_2,\mathscr{O}_X)\to\cdots.$$

By our assumption on $ch(E)$, we have

$$ch_i(H^{-1}(E_1)^*)=ch_i(H^0(E^\vee[2]))=ch_i(I_Z\otimes L)\text{ for } i=0,1,2.$$

Hence $ch(H^{-1}(E_1)^* \otimes L^*)$ is of the form $(1,0,0,*)$. Since $H^{-1}(E_1)^* \otimes L^*$ is a reflexive sheaf, by our assumption on X and [Sim, Theorem 2], this forces $H^{-1}(E_1)^* \otimes L^* \cong \mathscr{O}_X$. Hence $H^0(E^\vee[2]) = I_Y \otimes L$ for some 0-dimensional subscheme $Y \subset X$, while $H^0(E) = H^0(E_2) \in \mathrm{Coh}^{\leq 0}(X)$ as wanted. □

4. Tilt-semistable objects for $\omega \to \infty$

In [BMT, Section 7.2], Bayer-Macrì-Toda consider a subcategory $\mathfrak{D} \subset \mathcal{B}_{\omega,B}$ when ω is an ample $\mathbb{Q}$-divisor, where $\mathfrak{D}$ consists of objects $E \in \mathcal{B}_{\omega,B}$ of the following form:

(a) $H^{-1}(E) = 0$, and $H^0(E)$ is a pure sheaf of dimension ≥ 2 which is slope semistable with respect to ω.
(b) $H^{-1}(E) = 0$, and $H^0(E) \in \mathrm{Coh}^{\leq 1}(X)$.
(c) $H^{-1}(E)$ is a torsion-free slope semistable sheaf, and $H^0(E) \in \mathrm{Coh}^{\leq 1}(X)$; if $\mu_{\omega,B}(H^{-1}(E)) < 0$, then also $\mathrm{Hom}(\mathrm{Coh}^{\leq 1}(X), E) = 0$.

And we have:

Lemma 4.1. [BMT, Lemma 7.2.1] *If $E \in \mathcal{B}_{\omega,B}$ is $\nu_{m\omega,B}$-semistable for $m \gg 0$, then $E \in \mathfrak{D}$.*

Remark 4.2. We point out that any dual-PT-semistable complex (e.g. those termed as σ_3-semistable in [Lo2]) of positive degree is of type (c) in the category $\mathfrak{D}$ above. We do not know whether all dual-PT-semistable complexes of positive degree are $\nu_{m\omega,B}$-semistable for $m \gg 0$, although we take one step in this direction in Lemma 4.4 below.

In this section, we try to prove the converse of Lemma 4.1, which would give examples of tilt-stable objects when $\omega \to \infty$. Since tilt-semistable objects with $\nu_{\omega,B} = 0$ are $Z_{\omega,B}$-semistable objects of phase 1 in $\mathcal{A}_{\omega,B}$, these results can help us describe Bridgeland semistable objects on threefolds as $\omega \to \infty$.

To start with, we observe the following easy consequence of Lemma 4.1 and Theorem 3.10:

Lemma 4.3. *Suppose $E \in \mathcal{B}_{\omega,B}$ is such that $\overline{\Delta}_\omega(E) = 0$, $ch_0(E) < 0$, $c_1(E)$ is proportional to ω and $\omega^2\widetilde{ch}_1(H^{-1}(E)) < 0$. If E is $\nu_{m\omega,B}$-semistable for $m \gg 0$, then $E = H^{-1}(E)[1]$ where $H^{-1}(E)$ is a $\mu_{\omega,B}$-semistable sheaf.*

The next lemma is one step towards the converse of Lemma 4.1 for objects of type (c) above:

Lemma 4.4. *Suppose $E \in \mathcal{B}_{\omega,B}$ satisfies the following: $H^{-1}(E)$ is a torsion-free slope stable sheaf, $H^0(E) \in \mathrm{Coh}^{\leq 1}(X)$, $\mu_{\omega,B}(H^{-1}(E)) < 0$ and $\mathrm{Hom}(\mathrm{Coh}^{\leq 1}(X), E) = 0$. Then for any short exact sequence in $\mathcal{B}_{\omega,B}$*

$$0 \to M \to E \to N \to 0 \tag{4.1}$$

where $M, N \neq 0$, we have $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(N)$ for $m \gg 0$.

Note that, Lemma 4.4 does not necessarily imply E is $\nu_{m\omega,B}$-stable for $m \gg 0$, since m might depend on the particular short exact sequence (4.1) being considered. To show that such E is $\nu_{m\omega,B}$-stable for $m \gg 0$, one might need to bound the Chern classes of all the M or N that appear in such short exact sequences, as is done in [LQ, Theorem 1.1(ii)].

Before we prove Lemma 4.4, let us make some observations:

(i) The category $\mathcal{B}_{\omega,B}$ is invariant under replacing ω by $m\omega$ for any $m > 0$.
(ii) If A, C are two objects in $\mathcal{B}_{\omega,B}$ such that $\omega\widetilde{ch}_1(A), \widetilde{ch}_1(C) \neq 0$, then we have

$$-\frac{1}{\mu_{\omega,B}(A)} < -\frac{1}{\mu_{\omega,B}(C)} \text{ if and only if } \nu_{m\omega,B}(A) < \nu_{m\omega,B}(C) \text{ for } m \gg 0. \tag{4.2}$$

This is immediate from the equation

$$\nu_{m\omega,B}(-) = \frac{m\omega\widetilde{ch}_2(-) - \frac{m^3\omega^3}{6}\widetilde{ch}_0(-)}{m^2\omega^2\widetilde{ch}_1(-)}. \tag{4.3}$$

Proof of Lemma 4.4. Consider a short exact sequence (4.1) where $M, N \neq 0$. To show that $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(N)$ for $m \gg 0$, let us divide into two cases:
Case 1: $H^{-1}(M) \neq 0$. By the $\mu_{\omega,B}$-stability of $H^{-1}(E)$ and the assumption that $\mu_{\omega,B}(H^{-1}(E)) < 0$, we have $\omega^2\widetilde{ch}_1(H^{-1}(M)) < 0$. This implies $\omega^2\widetilde{ch}_1(M) > 0$, and so $\nu_{m\omega,B}(M) < +\infty$ for all $m > 0$. If $\omega^2\widetilde{ch}_1(N) = 0$, then $\nu_{m\omega,B}(N) = +\infty$ for all $m > 0$, and so we have $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(N)$ for all $m > 0$. For the remainder of Case 1, let us assume that $\omega^2\widetilde{ch}_1(N) \neq 0$. Consider the long exact sequence of (4.1):

$$0 \to H^{-1}(M) \overset{\alpha}{\to} H^{-1}(E) \overset{\beta}{\to} H^{-1}(N) \overset{\gamma}{\to} H^0(M) \overset{\delta}{\to} H^0(E) \to H^0(N) \to 0. \tag{4.4}$$

Suppose $\operatorname{im}\gamma = 0$. Then we have $\mu_{\omega,B}(H^{-1}(M)) < \mu_{\omega,B}(H^{-1}(N)) < 0$, implying $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(N)$ for $m \gg 0$ by (4.2). If $\operatorname{im}\gamma \neq 0$, then we have $\mu_{\omega,B}(H^{-1}(M)) < \mu_{\omega,B}(\operatorname{im}\beta)$ as well as

$$\mu_{\omega,B}(\operatorname{im}\beta) \leq \mu_{\omega,B}(H^{-1}(N)) \leq 0 \leq \mu_{\omega,B}(\operatorname{im}\gamma) \tag{4.5}$$

by the see-saw principle. Hence $\mu_{\omega,B}(H^{-1}(M)) < \mu_{\omega,B}(H^{-1}(N)) < 0$, and we have

$$\nu_{m\omega,B}(H^{-1}(M)) < \nu_{m\omega,B}(H^{-1}(N)) \text{ for } m \gg 0 \tag{4.6}$$

by (4.2). Note that both sides of (4.6) are $O(m)$ in magnitude.
Now, we have

$$\nu_{m\omega,B}(H^{-1}(N)) = \nu_{m\omega,B}(N) - \frac{m\omega\widetilde{ch}_2(H^0(N))}{m^2\omega^2\widetilde{ch}_1(H^{-1}(N)[1])}. \tag{4.7}$$

On the other hand,

$$\begin{aligned}
\nu_{m\omega,B}(M) &\leq \nu_{m\omega,B}(M) + \frac{\frac{m^3\omega^3}{6}\widetilde{ch}_0(H^0(M))}{m^2\omega^2\left(\widetilde{ch}_1(H^{-1}(M)[1]) + \widetilde{ch}_1(H^0(M))\right)} \\
&= \frac{m\omega\widetilde{ch}_2(M) - \frac{m^3\omega^3}{6}\widetilde{ch}_0(H^{-1}(M)[1])}{m^2\omega^2\left(\widetilde{ch}_1(H^{-1}(M)[1]) + \widetilde{ch}_1(H^0(M))\right)} \\
&\leq \frac{m\omega\widetilde{ch}_2(M)}{m^2\omega^2\left(\widetilde{ch}_1(H^{-1}(M)[1]) + \widetilde{ch}_1(H^0(M))\right)} - \frac{\frac{m^3\omega^3}{6}\widetilde{ch}_0(H^{-1}(M)[1])}{m^2\omega^2\widetilde{ch}_1(H^{-1}(M)[1])} \\
&= \frac{m\omega\widetilde{ch}_2(M)}{m^2\omega^2\left(\widetilde{ch}_1(H^{-1}(M)[1]) + \widetilde{ch}_1(H^0(M))\right)} \\
&\qquad - \frac{m\omega\widetilde{ch}_2(H^{-1}(M)[1])}{m^2\omega^2\widetilde{ch}_1(H^{-1}(M)[1])} + \nu_{m\omega,B}(H^{-1}(M)[1]).
\end{aligned}$$

Letting $m \to \infty$ in the above inequalities while noting $\nu_{m\omega,B}(H^{-1}(M)[1]) = \nu_{m\omega,B}(H^{-1}(M))$, together with (4.6) and (4.7), we obtain $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(N)$ for $m \gg 0$. This completes the proof of Case 1.

Case 2: $H^{-1}(M) = 0$. In this case, if $\operatorname{im}\gamma = 0$, then $M = H^0(M) \in \mathrm{Coh}^{\leq 1}(X)$, contradicting our assumption $\mathrm{Hom}(\mathrm{Coh}^{\leq 1}(X), E) = 0$. So suppose $\operatorname{im}\gamma \neq 0$.

If $\mathrm{rk}(H^0(M)) \neq 0$, then $\widetilde{ch}_1(H^0(M)) > 0$ by the definition of $\mathcal{T}_{\omega,B}$, and we have $\nu_{m\omega,B}(M) = \nu_{m\omega,B}(H^0(M)) < 0$ for $m \gg 0$ from (4.3), while $\nu_{m\omega,B}(N) > 0$ for $m \gg 0$. That is, $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(N)$ for $m \gg 0$. Now, suppose $\mathrm{rk}(H^0(M)) = 0$ instead.

If $\operatorname{im}\gamma \in \mathrm{Coh}^{\leq 1}(X)$, then since $H^{-1}(N)$ is torson-free, we obtain a nonzero class in $\mathrm{Ext}^1(\operatorname{im}\gamma, H^{-1}(E)) \cong \mathrm{Hom}(\operatorname{im}\gamma, H^{-1}(E)[1])$, again contradicting our assumption $\mathrm{Hom}(\mathrm{Coh}^{\leq 1}(X), E) = 0$.

If $\operatorname{im}\gamma$ is supported in dimension 2, then so is $H^0(M)$, and so $\nu_{m\omega,B}(M) = \nu_{m\omega,B}(H^0(M)) \to 0$ as $m \to \infty$, while $\nu_{m\omega,B}(N) > 0$ for $m \gg 0$ from (4.3). Hence $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(N)$ for $m \gg 0$. This completes Case 2. $\square$

The following lemma and corollary are more concrete than Lemma 4.4 - it tells us that line bundles are $\nu_{\omega,B}$-stable when $\omega \to \infty$:

Lemma 4.5. *Let E be a line bundle with $\omega^2\widetilde{ch}_1(E) < 0$. Then there exists a constant $m_0 > 0$, depending only on $c_1(E)$, such that $E[1]$ is $\nu_{m\omega,B}$-stable whenever $m > m_0$.*

Proof. To prove the lemma, it suffices to find a constant $m_0 > 0$, depending only on $ch(E)$, such that for every short exact sequence in $\mathcal{B}_{\omega,B}$

$$0 \to M \to E[1] \to N \to 0 \tag{4.8}$$

where M is a maximal destabilising subobject of $E[1]$ with respect to $\nu_{m\omega,B}$ for some $m > 0$, we have $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(E[1])$ for $m > m_0$.

The long exact sequence of cohomology of (4.8) is

$$0 \to H^{-1}(M) \xrightarrow{\alpha} E \xrightarrow{\beta} H^{-1}(N) \xrightarrow{\gamma} H^0(M) \to 0. \tag{4.9}$$

If $H^{-1}(M)$ is of rank 1, then β is the zero map, meaning $H^{-1}(N) \cong H^0(M)$. This forces $N = 0$, contradicting our assumption. Hence $H^{-1}(M)$ must be zero.

If $\omega^2\widetilde{ch}_1(M) = 0$, then $M = H^0(M)$ must lie in $\mathrm{Coh}^{\leq 1}(X)$, giving us a subobject of E that lies in $\mathrm{Coh}^{\leq 1}(X)$; this contradicts $\mathrm{Ext}^1(\mathrm{Coh}^{\leq 1}(X), E) = 0$. Hence $\omega^2\widetilde{ch}_1(M) > 0$. Then, since we are assuming M is destabilising, we have $\nu_{m\omega,B}(N) < \infty$, and so $\omega^2\widetilde{ch}_1(N) > 0$.

Since $M = H^0(M)$ is $\nu_{m\omega,B}$-semistable for some m, by [BMT, Corollary 7.3.2] we have $\overline{\Delta}_{m\omega}(H^0(M)) \geq 0$, i.e.

$$(\omega^2\widetilde{ch}_1(H^0(M)))^2 \geq 2\omega^3\widetilde{ch}_0(H^0(M))\omega\widetilde{ch}_2(H^0(M)),$$

which gives

$$\frac{\omega\widetilde{ch}_2(H^0(M))}{\omega^2\widetilde{ch}_1(H^0(M))} \leq \frac{\omega^2\widetilde{ch}_1(H^0(M))}{2\omega^3\widetilde{ch}_0(H^0(M))} = \frac{1}{2\omega^3}\mu_\omega(H^0(M)). \tag{4.10}$$

On the other hand, if we let $\delta = \widetilde{ch}_1(E)$, then since $\widetilde{ch}_1(H^{-1}(N)) = \delta + \widetilde{ch}_1(H^0(M))$, we have

$$0 < \omega^2\widetilde{ch}_1(H^0(M)) = \omega^2\widetilde{ch}_1(H^{-1}(N)) - \omega^2\delta < -\omega^2\delta. \tag{4.11}$$

Combining this with (4.10), we get

$$\frac{\omega\widetilde{ch}_2(H^0(M))}{\omega^2\widetilde{ch}_1(H^0(M))} < -\frac{\omega^2\delta}{2\omega^3}. \tag{4.12}$$

Hence, when $m \geq 1$, we have

$$\begin{aligned}
\nu_{m\omega,B}(M) &= \nu_{m\omega,B}(H^0(M)) \\
&= \frac{m\omega\widetilde{ch}_2(H^0(M)) - \frac{m^3\omega^3}{6}\widetilde{ch}_0(H^0(M))}{m^2\omega^2\widetilde{ch}_1(H^0(M))} \\
&< -\frac{\omega^2\delta}{m2\omega^3} - m\frac{\omega^3\widetilde{ch}_0(H^0(M))}{6\omega^2\widetilde{ch}_1(H^0(M))} \\
&\leq -\frac{\omega^2\delta}{2\omega^3} - m\frac{\omega^3\widetilde{ch}_0(H^0(M))}{6\omega^2\widetilde{ch}_1(H^0(M))} \\
&\leq -\frac{\omega^2\delta}{2\omega^3}.
\end{aligned} \tag{4.13}$$

Since

$$\nu_{m\omega,B}(E) = \frac{m\omega\widetilde{ch}_2(E) - \frac{m^3\omega^3}{6}\widetilde{ch}_0(E)}{m^2\omega^2\widetilde{ch}_1(E)},$$

it is clear that, there is a constant $m_0 > 0$ depending only on $ch(E)$, hence only on $c_1(E)$, such that $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(E[1])$ whenever $m > m_0$. This implies that $\nu_{m\omega,B}(M) < \nu_{m\omega,B}(N)$ whenever $m > m_0$, i.e. $E[1]$ is $\nu_{m\omega,B}$-stable whenever $m > m_0$. □

The following proposition computes an explicit bound for m_0 that appeared in Lemma 4.5. Part (c) of the proposition can also be used to verify the inequality in Conjecture 2.2:

Proposition 4.6. *Let (X, ω) be a polarised smooth projective threefold, and $m > 0$. Suppose $B = 0$, E is a line bundle on X, and let $d := c_1(E)\omega^2 < 0$. Then for $m > 0$,*

(a) $\nu_{m\omega,B}(E[1]) = 0$ *if and only if* $m^2 = \frac{3c_1(E)^2\omega}{\omega^3}$.

(b) *If* $\nu_{m\omega,B}(E[1]) = 0$*, then* $\nu_{m\omega,B}(E[1])$ *is* $\nu_{m\omega,B}$*-stable whenever* $m^2 \geq \frac{3d^2}{(\omega^3)^2}$.

(c) $\widetilde{ch}_3(E[1]) < \frac{m^2\omega^2}{2}\widetilde{ch}_1(E[1])$ *is equivalent to* $m^2 > \frac{c_1(E)^3}{3d}$.

Note that, if $c_1(E)$ is proportional to ω, then $\overline{\Delta}_\omega(E) = 0$, in which case equality holds in Conjecture 2.2 by results in [BMT, Section 7.4].

Proof. (a) That $\nu_{m\omega,B}(E[1]) = 0$ is equivalent to

$$m\omega ch_2(E[1]) = \frac{m^3\omega^3}{6}ch_0(E[1]),$$

i.e. $m^2\omega^3 = 3c_1^2\omega$, and so the claim follows.

(b) Suppose $\nu_{m\omega,B}(E[1]) = 0$. From the proof of Lemma 4.5, it suffices to show

$$-\frac{d}{m2\omega^3} - m\frac{\omega^3 ch_0(H^0(M))}{6\omega^2 ch_1(H^0(M))} \leq 0. \tag{4.14}$$

whenever $m^2 \geq \frac{3d^2}{(\omega^3)^2}$, where M is as in the inequalities (4.13).

Now, from (4.11) we have

$$\frac{1}{\omega^2 ch_1(H^0(M))} > -\frac{1}{d},$$

and hence

$$\begin{aligned} -\frac{d}{m2\omega^3} - m\frac{\omega^3 ch_0(H^0(M))}{6\omega^2 ch_1(H^0(M))} &< -\frac{d}{m2\omega^3} + m\frac{\omega^3 ch_0(H^0(M))}{6d} \\ &\leq -\frac{d}{m2\omega^3} + \frac{m\omega^3}{6d}. \end{aligned}$$

Therefore, (4.14) holds if $-\frac{d}{m2\omega^3} + \frac{m\omega^3}{6d} \leq 0$, which is equivalent to $m^2 \geq \frac{3d^2}{(\omega^3)^2}$, and the claim follows.

(c) That $\widetilde{ch}_3(E[1]) < \frac{m^2\omega^2}{2}\widetilde{ch}_1(E[1])$ is equivalent to $-\frac{c_1(E)^3}{6} < -\frac{m^2\omega^2 c_1(E)}{2}$, i.e. $c_1(E)^3 > 3dm^2$. Since $d < 0$, this is equivalent to $m^2 > \frac{c_1(E)^3}{3d}$ as claimed. ∎

5. Objects with twice minimal $\omega^2 ch_1$

In [BMT, Lemma 7.2.2], tilt-semistable objects F with $\omega^2\widetilde{ch}_1(F) \leq c$ are characterised, where

$$c := \min\{\omega^2\widetilde{ch}_1(F) > 0 \mid F \in \mathcal{B}_{\omega,B}\}. \tag{5.1}$$

In the next proposition, we give some sufficient conditions for a torsion-free sheaf $E \in \mathcal{T}_{\omega,B}$ with $\omega^2\widetilde{ch}_1(E) = 2c$ to be tilt-stable.

Proposition 5.1. *Suppose* $E \in \mathcal{T}_{\omega,B}$ *is a torsion-free sheaf with* $\nu_{\omega,B}(E) = 0$ *and* $\omega^2\widetilde{ch}_1(E) = 2c$*, where* c *is defined in* (5.1).

(1) *If* $\mu_{\omega,B,\max}(E) < \frac{\omega^3}{\sqrt{3}}$*, then* E *is* $\nu_{\omega,B}$*-stable.*

(2) *If* $\omega^3 > 3\omega(\widetilde{ch}_1(M))^2$ *for every torsion free slope semistable sheaf* M *with* $\omega^2\widetilde{ch}_1(M) = c$*, then* E *is* $\nu_{\omega,B}$*-stable.*

Proof. Suppose we have a destabilizing short exact sequence in $\mathcal{B}_{\omega,B}$

$$0 \to M \to E \to N \to 0 \tag{5.2}$$

with $\nu_{\omega,B}(M) \geq \nu_{\omega,B}(E) = 0 \geq \nu_{\omega,B}(N)$, and we may assume M is $\nu_{\omega,B}$-stable by replacing it with its maximal destabilizing subobject in $\mathcal{B}_{\omega,B}$ with respect to $\nu_{\omega,B}$-stability. The long exact sequence associated to (5.2) is

$$0 \to H^{-1}(N) \xrightarrow{\alpha} H^0(M) \xrightarrow{\beta} E \xrightarrow{\gamma} H^0(N) \to 0 \tag{5.3}$$

and we identify $M = H^0(M)$.

Since $\omega^2\widetilde{ch}_1(E) = 2c$, the possibilities for $(\omega^2\widetilde{ch}_1(M), \omega^2\widetilde{ch}_1(N))$ are $(2c, 0)$, (c, c), and $(0, 2c)$. The cases $(2c, 0)$ and $(0, 2c)$ are easily eliminated as possibilities as follows:

- case $(0, 2c)$: Since $M = H^0(M) \in \mathcal{T}_{\omega,B}$, the condition $\omega^2\widetilde{ch}_1(M) = 0$ forces M to be torsion. Since E is torsion free, we have $\beta = 0$ in (5.3), hence $H^{-1}(N) \cong H^0(M)$, which forces $M = H^0(M) = 0$, contrary to assumption.
- case $(2c, 0)$: in this case $\nu_{\omega,B}(N) = \infty$ and equation (5.3) cannot be a destabilizing sequence.

We now consider the case (c, c). Since M is $\nu_{\omega,B}$-stable with $\omega^2\widetilde{ch}_1(M) = c$, by [BMT, Lemma 7.2.2] we know that $H^0(M)$ lies in the set $\mathfrak{D}$ described in [BMT, Section 7.2]. Since $H^{-1}(N)$ and E are torsion free, we have M is torsion free, and by the description of elements of $\mathfrak{D}$ we have that M is a torsion-free slope semistable sheaf.

Since M is $\nu_{\omega,B}$-stable, by [BMT, Cor. 7.3.2] we also have

$$\omega\widetilde{ch}_2(M) \leq \frac{(\omega^2\widetilde{ch}_1(M))^2}{2\omega^3\widetilde{ch}_0(M)}. \tag{5.4}$$

Since $\nu_{\omega,B}(E) = 0$, the inequality $\nu_{\omega,B}(M) \geq 0$ implies

$$\omega\widetilde{ch}_2(M) \geq \frac{\omega^3}{6}\widetilde{ch}_0(M). \tag{5.5}$$

Combining equation (5.5) with equation (5.4) we get

$$\frac{\omega^3}{6}\widetilde{ch}_0(M) \leq \frac{(\omega^2\widetilde{ch}_1(M))^2}{2\omega^3\widetilde{ch}_0(M)} \tag{5.6}$$

or $\frac{\omega^3}{\sqrt{3}} \leq \mu_{\omega,B}(M)$.

The hypothesis $\mu_{\omega,B,\max}(E) < \frac{\omega^3}{\sqrt{3}}$ implies $\mu_{\omega,B,\max}(E) < \mu_{\omega,B}(M)$. Since M is slope semistable, this inequality implies $\mathrm{Hom}_{\mathrm{Coh}(X)}(M, E) = 0$ and hence $\beta = 0$ in (5.3). We then get a contradiction as in the $(0, 2c)$ case. This completes the proof of part (1).

To prove part (2), suppose E has a destabilising subobject M as in (5.2). The usual Bogomolov-Giesker inequality gives us

$$\omega\widetilde{ch}_2(M) \leq \frac{\omega(\widetilde{ch}_1(M))^2}{2\widetilde{ch}_0(M)}. \tag{5.7}$$

Combining (5.7) with (5.5), we get

$$\frac{\omega^3}{6}\widetilde{ch}_0(M) \le \omega\widetilde{ch}_2(M) \le \frac{\omega(\widetilde{ch}_1(M))^2}{2\widetilde{ch}_0(M)}, \tag{5.8}$$

and hence $\omega^3(\widetilde{ch}_0(M))^2 \le 3\omega(\widetilde{ch}_1(M))^2$. Since M is a torsion-free sheaf, we have $\widetilde{ch}_0(M) \ge 1$, and hence $\omega^3 \le 3\omega(\widetilde{ch}_1(M))^2$. Part (2) thus follows. □

In [BMT, Example 7.2.4], Conjecture 2.2 was studied for rank-one sheaves of the form $E = L \otimes I_C$, where L is a line bundle, I_C the ideal sheaf of a curve on X, and $\omega^2 c_1(E) = c$. In the next proposition, following the ideas in [Tod, Remark 2.10], we study rank-one sheaves of the form $E = L^2 \otimes I_C$ where $\omega^2 c_1(E) = 2c$. In particular, we apply Proposition 5.1 to find a condition when E is $\nu_{\omega,B}$-semistable. In part (4) of the proposition, we are able to verify Conjecture 2.2 for these particular objects E by reducing the conjecture to the classical Castelnuovo inequality.

Proposition 5.2. *Let $B = 0$. Suppose $\mathrm{Pic}(X)$ is generated by an ample line bundle L on X. Let $h := c_1(L)$, $D := h^3$, and $\omega := mh$ for some positive $m \in \mathbb{Q}$. Suppose $C \subset X$ be a curve in X of degree $d := h \cdot [C] = h \cdot ch_2(\mathscr{O}_C)$. Let I_C be the ideal sheaf of $C \subset X$, and let*

$$E := L^2 \otimes I_C.$$

1. *If $\nu_{\omega,0}(E) = 0$ then $m^2 = 12 - \frac{6d}{D}$ and $d < 2D$. The converse also holds.*
2. *If $\nu_{\omega,0}(E) = 0$ and $d < \frac{3}{2}D$, then E is $\nu_{\omega,0}$-stable.*
3. *If $-ch_3(\mathscr{O}_C) \le \frac{4}{3}d$ and $\nu_{\omega,0}(E) = 0$ then E satisfies the inequality in Conjecture 2.2.*
4. *If $d \le D$, and $\nu_{\omega,0}(E) = 0$, and $X \subset \mathbb{P}^4$ is a hypersurface of degree D, then E satisfies the inequality in Conjecture 2.2.*

Proof. We follow the argument in [BMT, Example 7.2.4]. To start with, note that

$$\begin{aligned} ch_1(E) &= 2h, \\ ch_2(E) &= ch_0(L^2)ch_2(I_C) + ch_1(L^2)ch_1(I_C) + ch_2(L^2)ch_0(I_C) = -[C] + 2h^2, \text{and} \\ ch_3(E) &= ch_3(L^2)ch_0(I_C) + ch_2(L^2)ch_1(I_C) + ch_1(L^2)ch_2(I_C) + ch_0(L^2)ch_3(I_C) \\ &= \frac{4D}{3} - 2d - ch_3(O_C). \end{aligned}$$

For part (1), note that $\nu_{\omega,0}(E) = 0$ is equivalent to $mh \cdot ch_2(E) = \frac{m^3h^3}{6}$, i.e. $-d + 2D = \frac{m^2D}{6}$, i.e. $m^2 = 12 - \frac{6d}{D}$. Since $m^2 > 0$, it follows that $d < 2D$.

To prove part (2), we use of Proposition 5.1. In our situation, $c = \omega^2 h = m^2h^3$. Take any torsion-free slope semistable sheaf M with $\omega^2 ch_1(M) = c$. Then $ch_1(M) = h$. By part (2) of Proposition 5.1, E would be $\nu_{\omega,B}$-stable if we can show $\omega^3 > 3\omega(ch_1(M))^2$ i.e. $m^3h^3 > 3mh(h^2)$, or $m^2 > 3$; since $m^2 = 12 - \frac{6d}{D}$, this is equivalent to $d < \frac{3}{2}D$.

For part (3), just note that the inequality in Conjecture 2.2 now reads

$$\frac{4D}{3} - 2d - ch_3(O_C) \le \frac{m^2 \cdot 2D}{18} = \frac{4D}{3} - \frac{2d}{3} \tag{5.9}$$

or, equivalently,

$$-ch_3(\mathscr{O}_C) \le \frac{4}{3}d. \tag{5.10}$$

(Note that this is a stronger requirement than [BMT, Equation (32)].)

For part (4), if $X \subset \mathbb{P}^4$ is a hypersurface of degree D, then by Hirzebruch-Riemann-Roch we have

$$1 - g = \chi(\mathscr{O}_C) = ch_3(\mathscr{O}_C) + \frac{d}{2}(5 - D).$$

Then (5.9) becomes

$$g \leq \frac{dD}{2} - \frac{7}{6}d + 1. \tag{5.11}$$

When $d \leq D$, the bound on g in equation 5.11 follows from the Castelnuovo inequality $g \leq \frac{1}{2}(d-1)(d-2)$. (However in general, we only know $d < 2D$, and also we do not know if E is $\nu_{\omega,0}$-stable.) □

6. Tilt-unstable objects

In this section, we use known inequalities between Chern characters of reflexive sheaves on $\mathbb{P}^3$ to describe many slope stable reflexive sheaves $E \in \mathcal{B}_{\omega,B}$ that are tilt-unstable. We base our examples on the following result of Miró-Roig:

Proposition 6.1. [Mir, Prop. 2.18] *Let $X = \mathbb{P}^3$ and $B = 0$. For all c_2, c_3 such that $c_2 \geq 3$, c_3 is even, and $-c_2^2 + c_2 \leq c_3 \leq 0$, there exists a rank 3 stable reflexive sheaf on $\mathbb{P}^3$ with first through third Chern classes $(0, c_2, c_3)$.*

Proposition 6.2. *Let $X = \mathbb{P}^3$, $\omega = c_1(\mathscr{O}(1))$, and $B = 0$. Let n and m be positive integers of the same parity and with $3n^2 - m^2 \geq 6$. Let E be a slope stable rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1(E) = 0$, $c_2(E) = \frac{3n^2-m^2}{2}$ and $c_3(E)$ an even integer satisfying*

$$-(2n^3 + \frac{2nm^2}{3}) > c_3(E) \geq -\frac{(9n^4 - 6n^2m^2 + m^4 - 6n^2 + 2m^2)}{4}$$

(such an E exists by Proposition 6.1). Let $F = E(-n)[1]$. Then F is tilt-unstable.

Proof. The condition $\nu_{m\omega,0}(F) = 0$, i.e. $ch_2(F) = \frac{m^2}{6}ch_0(F)$, is equivalent to $c_2(E) = \frac{3n^2-m^2}{2}$.

First, note that such an E exists: to use Miro-Roig's result, we require $0 \geq c_3 \geq -c_2^2 + c_2$ and $c_2 \geq 3$. If $c_2 = \frac{3n^2-m^2}{2}$, the first inequality becomes $0 \geq c_3 \geq -\frac{(9n^4-6n^2m^2+m^4-6n^2+2m^2)}{4}$, and the second becomes $3n^2 - m^2 \geq 6$. So such an E exists. Let $F = E(-n)[1]$. Then $ch_3(F) = \frac{n^3}{2} - nc_2 - \frac{c_3}{2}$ where $c_i = c_i(E)$, and we have $\mu_{\omega,B}(F) = \frac{c_1-3n}{3} = -n$. Since $n > 0$, we have $F \in \mathcal{B}_{\omega,B}$.

Now, we claim that $ch_3(F) > \frac{m^2}{18}ch_1(F)$. Observe that

$$\begin{aligned} ch_3(F) &> \frac{m^2}{18}ch_1(F) \\ \Leftrightarrow \frac{n^3}{2} - nc_2 - \frac{c_3}{2} &> \frac{3nm^2}{18} \\ \Leftrightarrow \frac{n^3}{2} - n\frac{(3n^2-m^2)}{2} - \frac{c_3}{2} &> \frac{3nm^2}{18} \\ \Leftrightarrow -(2n^3 + \frac{2nm^2}{3}) &> c_3(E), \end{aligned}$$

which holds by assumption. Since Conjecture 2.2 holds on $\mathbb{P}^3$, as is proved in [Mac], F must be $\nu_{\omega,0}$-unstable. □

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